

Eigenvalue problem for a p-Laplacian equation with trapping potentials

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Abstract

Consider the following eigenvalue problem of p-Laplacian equation

$$-\Delta_p u + V(x)|u|^{p-2}u = \mu|u|^{p-2}u + a|u|^{s-2}u, x \in \mathbb{R}^n, \quad (\text{P})$$

where $a \geq 0$, $p \in (1, n)$ and $\mu \in \mathbb{R}$. $V(x)$ is a trapping type potential, e.g., $\inf_{x \in \mathbb{R}^n} V(x) < \lim_{|x| \rightarrow +\infty} V(x)$. By using constrained variational methods, we proved that there is $a^* > 0$, which can be given explicitly, such that problem (P) has a ground state u with $\|u\|_{L^p} = 1$ for some $\mu \in \mathbb{R}$ and all $a \in [0, a^*)$, but (P) has no this kind of ground state if $a \geq a^*$. Furthermore, by establishing some delicate energy estimates we show that the global maximum point of the ground states of problem (P) approach to one of the global minima of $V(x)$ and blow up if $a \nearrow a^*$. The optimal rate of blowup is obtained for $V(x)$ being a polynomial type potential.

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1 Introduction

In this paper, we are concerned with the existence and asymptotical behavior of ground states for the following eigenvalue problem of p-Laplacian equation:

$$-\Delta_p u + V(x)|u|^{p-2}u = \mu|u|^{p-2}u + a|u|^{s-2}u, x \in \mathbb{R}^n, \quad (1.1)$$

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where $p \in (1, n)$, $s = p + \frac{p^2}{n}$, $a \geq 0$ and $\mu \in \mathbb{R}$ are parameters, $V(x)$ is a trapping potential which satisfies

$$(V) : V(x) \in C(\mathbb{R}^n), \lim_{|x| \rightarrow \infty} V(x) = \infty \text{ and } \inf_{x \in \mathbb{R}^n} V(x) = 0.$$

When $p = n = 2$, (1.1) is the so called time independent Gross-Pitaevskii (GP) equation, which was proposed independently by Gross[8] and Pitaevskii[20] in studying the Bose-Einstein condensate. In this special case, problem (1.1) has been studied under various conditions on the potential $V(x)$, for examples, [10, 11, 12], etc. Roughly speaking, if $V(x)$ is a polynomial type trapping potential such as

$$V(x) = h(x) \prod_{i=1}^m |x - x_i|^{q_i}, x_i \neq x_j \text{ if } i \neq j, 0 < C \leq h(x) \leq \frac{1}{C} \text{ for all } x \in \mathbb{R}^n, \quad (1.2)$$

the results of [10] show that the existence of normalized L^2 -norm ground states of (1.1) depends heavily on the parameter $a \geq 0$, and this kind of solution blows up at some point x_{i_0} with $q_{i_0} = \max\{q_1, \dots, q_m\}$. The rate of blowup is also given in [10]. The main aim of this paper is to extend the results of [10] to the p-Laplacian problem (1.1) for general $p \in (1, n)$ and $V(x)$.

As we know, the operator $-\Delta_p$ is no more linear if $p \neq 2$, which leads to some quite different properties from $-\Delta$ (i.e. $p = 2$), for examples, it is well known that the limit equation of (1.1), that is,

$$-\Delta_p u + \frac{p}{n} |u|^{p-2} u = |u|^{s-2} u, \quad (1.3)$$

has a unique positive radially symmetric solution (see e.g., [7, 13, 17, 19]) for $p = 2$, but in general case we know that this fact holds only for $p \in (1, 2)$ (see e.g., [6, 16, 23]), which is still unknown if $p \in (1, n)$ and $p \neq 2$. However, in [10] the uniqueness of solutions of the limit equation of (1.1) plays a crucial role not only in applying the Gagliardo-Nirenberg inequality, but also in getting the exact blowup rate for the ground state of (1.1). In order to extend the results of [10] to the p-Laplacian case, the key step is how to avoid using the uniqueness of solutions of the limit equation (1.3). In this paper, we overcome this difficulty by detailed analyzing the relations between the extremal functions of the sharp constant of the Gagliardo-Nirenberg inequality and the ground states of the limit equation (1.3). On the other hand, if $p \neq 2$, the expansion of the main part of the variational functional of (1.1) is also more complicated than that of $p = 2$, which causes more difficulties in making the energy estimates than in [10].

To get a ground state solution of (1.1), we consider the following constrained minimization problem

$$e(a) = \inf \{E_a(u) : u \in \mathcal{H}, \int_{\mathbb{R}^n} |u|^p dx = 1\}, \quad (1.4)$$

where E_a is the energy functional defined by

$$\begin{aligned} E_a(u) &= \int_{\mathbb{R}^n} (|\nabla u|^p + V(x)|u|^p) dx - \frac{pa}{s} \int_{\mathbb{R}^n} |u|^s dx \\ &= \int_{\mathbb{R}^n} (|\nabla u|^p + V(x)|u|^p) dx - \frac{na}{n+p} \int_{\mathbb{R}^n} |u|^s dx, \end{aligned} \quad (1.5)$$

for $u \in \mathcal{H} := \left\{ u \in W^{1,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^p dx < +\infty \right\}$ and

$$\|u\|_{\mathcal{H}} := \left(\int_{\mathbb{R}^n} |\nabla u|^p + V(x)|u(x)|^p dx \right)^{\frac{1}{p}}.$$

Clearly, a minimizer of (1.4) is a weak solution of (1.1) for some $\mu \in \mathbb{R}$, which is indeed a Lagrange multiplier.

For problem (1.4), the power $s = p + \frac{p^2}{n}$ is critical in the sense that $e(a)$ can be $-\infty$ if $s > p + \frac{p^2}{n}$. Indeed, take $u \in \mathcal{H}$ and $\|u\|_{L^p(\mathbb{R}^n)} = 1$ and let $v_\lambda(x) = \lambda^{\frac{n}{p}} u(\lambda x)$, it is easy to see that

$$E_a(v_\lambda) = \lambda^p \int_{\mathbb{R}^n} |\nabla u|^p dx + \int_{\mathbb{R}^n} V\left(\frac{x}{\lambda}\right) |u|^p dx - \frac{pa}{s} \lambda^{\frac{ns}{p}-n} \int_{\mathbb{R}^n} |u|^s dx \xrightarrow{\lambda \rightarrow \infty} -\infty,$$

if $\frac{ns}{p} - n > p$, i.e., $s > p + \frac{p^2}{n}$. When $p = n = 2$, $s = 4$ is the so called mass critical exponent for GP equation.

For $s = p + \frac{p^2}{n}$, we recall some known results about the limit equation (1.3). First we define the energy functional

$$I(u) = \frac{1}{p} \int_{\mathbb{R}^n} (|\nabla u|^p + \frac{p}{n} |u|^p) dx - \frac{1}{s} \int_{\mathbb{R}^n} |u|^s dx, u \in W^{1,p}(\mathbb{R}^n).$$

It is well known that u is a weak solution of (1.3) if and only if

$$\langle I'(u), \varphi \rangle = 0 \text{ for all } \varphi \in W^{1,p}(\mathbb{R}^n).$$

Next, we denote the set of all nontrivial weak solutions of (1.3) by \mathcal{S} , that is

$$\mathcal{S} := \{u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\} : \langle I'(u), \varphi \rangle = 0 \quad \forall \varphi \in W^{1,p}(\mathbb{R}^n)\}.$$

Then, for any $u \in \mathcal{S}$, by using Pohozaev identity(see [9]) we know that

$$\int_{\mathbb{R}^n} |u|^s dx = \left(1 + \frac{p}{n}\right) \int_{\mathbb{R}^n} |\nabla u|^p dx = \left(1 + \frac{p}{n}\right) \int_{\mathbb{R}^n} |u|^p dx, \quad (1.6)$$

where $s = p + \frac{p^2}{n}$. Now, we say $Q \in W^{1,p}(\mathbb{R}^n)$ is a *ground state* of (1.3) if it is the *least energy solution* among all nontrivial weak solutions of (1.3). Then, it follows from (1.6) that

$$Q \in \mathcal{G} := \{u \in \mathcal{S} : I(u) = \inf_{v \in \mathcal{S}} I(v)\} = \{u \in \mathcal{S} : I(u) = \inf_{v \in \mathcal{S}} \frac{1}{n} \int_{\mathbb{R}^n} |v|^p dx\}. \quad (1.7)$$

Clearly, if $Q(x) \in \mathcal{G}$, then $Q(x - x_0) \in \mathcal{G}$ for any $x_0 \in \mathbb{R}^n$. Furthermore, by the result of [16], any ground state of (1.1) decays exponentially at infinity, that is, for any $Q \in \mathcal{G}$ there exists $\delta > 0$ such that

$$|Q(x)| \leq e^{-\delta|x|}, \text{ for } |x| \text{ large.} \quad (1.8)$$

Finally, we give the main theorems of the paper. Our first theorem is concerned with the existence of minimizers of the minimization problem (1.4) and hence Lemma 2.4 implies the existence of ground states of (1.1), which is consistent with the results of [3, 10, 26] if $p = 2$.

Theorem 1.1 *Let $Q \in \mathcal{G}$ and let*

$$a^* = \left(\int_{\mathbb{R}^n} |Q|^p \right)^{\frac{p}{n}}. \quad (1.9)$$

If $p \in (1, n)$ and $V(x)$ satisfies the condition (V). Then,

- (i) *Problem (1.4) has at least one minimizer if $0 \leq a < a^*$.*
- (ii) *Problem (1.4) has no minimizer if $a \geq a^*$ and $e(a) = -\infty$ if $a > a^*$. Moreover, $e(a) > 0$ if $a < a^*$ and $\lim_{a \nearrow a^*} e(a) = e(a^*) = 0$.*

Remark 1.1 *The number a^* defined in (1.9) is independent of the choice of $Q \in \mathcal{G}$. In fact, let c_0 be the least energy of (1.3), then, for any $Q \in \mathcal{G}$, $I(Q) = c_0$ and it follows from (1.7) that $\int_{\mathbb{R}^n} |Q|^p dx = nc_0$, which is independent of $Q \in \mathcal{G}$.*

By Theorem 1.1, we know that, for any $a \in [0, a^*)$, problem (1.4) has a solution u_a , then it is interesting to ask what would happen when a goes to a^* from below, which is simply denoted by $a \nearrow a^*$ in what follows. Our next theorem answers this question for the general type of trapping potential $V(x)$ as in (V).

Theorem 1.2 *Let $u_a \geq 0$ be a minimizer of (1.4) for $a \in (0, a^*)$. If the condition (V) holds, then*

- (i)
$$\varepsilon_a \triangleq \left(\int_{\mathbb{R}^n} |\nabla u_a|^p \right)^{-\frac{1}{p}} \rightarrow 0 \quad \text{as } a \nearrow a^*. \quad (1.10)$$

- (ii) *Let \bar{z}_a be a global maximum point of $u_a(x)$, there holds*

$$\lim_{a \nearrow a^*} \text{dist}(\bar{z}_a, \mathcal{A}) = 0, \quad (1.11)$$

where $\mathcal{A} = \{x \in \mathbb{R}^n : V(x) = 0\}$.

(iii) For any sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \rightarrow \infty$, there exists a subsequence of $\{a_k\}$, still denoted by $\{a_k\}$, such that

$$\lim_{k \rightarrow \infty} \varepsilon_{a_k}^{\frac{n}{p}} u_{a_k}(\varepsilon_{a_k} x + \bar{z}_{a_k}) = \frac{Q(x)}{a^{*\frac{n}{p^2}}} \text{ in } W^{1,p}(\mathbb{R}^n), \text{ for some } Q \in \mathcal{G}, \quad (1.12)$$

where \bar{z}_{a_k} is a global maximum point of u_{a_k} and $\lim_{k \rightarrow \infty} \bar{z}_{a_k} = x_0 \in \mathcal{A}$.

The above theorem tells us that as $a \nearrow a^*$, the minimizers of (1.4) must concentrate and blow up at a minimum point of $V(x)$. Our final result shows that the concentration behavior and blow up rate of the minimizers of (1.4) can be refined if we have more information on the potential $V(x)$. Accurately, we assume that the trapping potential $V(x)$ is of some “polynomial type” and has $m \geq 1$ isolated minima, for instance, $V(x)$ is given by (1.2). Let $Q \in \mathcal{G}$ be given in Theorem 1.2, and let $y_0 \in \mathbb{R}^n$ be such that

$$\int_{\mathbb{R}^n} |x + y_0|^q Q^p(x) dx = \inf_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |x + y|^q Q^p(x) dx, \quad q = \max\{q_1, \dots, q_m\}. \quad (1.13)$$

Set

$$\lambda_i = \int_{\mathbb{R}^n} |x + y_0|^q Q^p(x) dx \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \in (0, \infty],$$

and

$$\lambda = \min\{\lambda_1, \dots, \lambda_m\}, \quad \mathcal{Z} := \{x_i : \lambda_i = \lambda\}. \quad (1.14)$$

Remark 1.2 If $1 < p \leq 2$, the ground state Q of (1.3) is unique (up to translation) and radially symmetric, see e.g., [23, 6, 7, 13, 17, 19], then it is not difficult to know that $y_0 = 0$ in (1.13).

Based on Theorem 1.2 and the above notations, we have the following theorem, which is a refined version of Theorem 1.2 when the potential $V(x)$ is given by (1.2).

Theorem 1.3 If $V(x)$ satisfies (1.2), let $\{a_k\} \subset (0, a^*)$ be the convergent subsequence in Theorem 1.2 (iii) and let u_{a_k} be a corresponding minimizer of (1.4), then

(i) For $e(a_k)$ defined by (1.4), there holds

$$e(a_k) \approx \frac{(a^* - a_k)^{\frac{q}{p+q}}}{a^{*\frac{n+q}{p+q}}} \lambda^{\frac{p}{p+q}} \left(\left(\frac{q}{p} \right)^{\frac{p}{p+q}} + \left(\frac{p}{q} \right)^{\frac{q}{p+q}} \right) \text{ as } k \rightarrow \infty, \quad (1.15)$$

where $f(a_k) \approx g(a_k)$ means that $f/g \rightarrow 1$ as $k \rightarrow \infty$.

(ii) Let $Q \in \mathcal{G}$ be obtained in (1.12) and let ε_{a_k} be defined by (1.10), then (1.12) still holds for u_{a_k} , but ε_{a_k} can be precisely estimated as

$$\varepsilon_{a_k} \approx \sigma_k \triangleq a^{*\frac{n-p}{p(p+q)}} (a^* - a_k)^{\frac{1}{p+q}} \lambda^{-\frac{1}{p+q}} \left(\frac{p}{q}\right)^{\frac{1}{p+q}}, \quad (1.16)$$

that is,

$$\lim_{k \rightarrow \infty} \sigma_k^{\frac{n}{p}} u_{a_k}(\sigma_k x + \bar{z}_{a_k}) = \frac{Q(x)}{a^{*\frac{n}{p^2}}} \text{ in } W^{1,p}(\mathbb{R}^n). \quad (1.17)$$

Moreover, for each k , if \bar{z}_{a_k} is a global maximum point of u_{a_k} , then

$$\lim_{k \rightarrow \infty} \bar{z}_{a_k} = x_0 \text{ with } x_0 \in \mathcal{Z}.$$

2 Preliminary lemmas

In this section, we give some useful lemmas which are required in next section.

Lemma 2.1 (*Gagliardo-Nirenberg inequality*) Let $p \in (1, n)$, $s = p + \frac{p^2}{n}$ and a^* be given by (1.9). Then, for any $u \in W^{1,p}(\mathbb{R}^n)$, there holds

$$\int_{\mathbb{R}^n} |u(x)|^s dx \leq \frac{n+p}{na^*} \int_{\mathbb{R}^n} |\nabla u(x)|^p dx \cdot \left(\int_{\mathbb{R}^n} |u(x)|^p dx \right)^{\frac{n}{n+p}}. \quad (2.1)$$

Moreover, the equality holds if and only if $u(x) = c_1 Q(c_2 x)$ for some $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ and $Q \in \mathcal{G}$.

Proof. By using Theorem 2.1 of [2] with $q = p$ and $s = p + \frac{p^2}{n}$, we see that

$$\int_{\mathbb{R}^n} |u(x)|^s \leq \left(\frac{K}{E(u_\infty)} \right)^{\frac{n+p}{n}} \int_{\mathbb{R}^n} |\nabla u(x)|^p \cdot \left(\int_{\mathbb{R}^n} |u(x)|^p \right)^{\frac{n}{n+p}}, \quad (2.2)$$

where $K = \frac{n+p}{np(\frac{p}{n})^{\frac{n+p}{n}}}$ and u_∞ is a minimizer of the following constrained minimization problem:

$$\inf \left\{ E(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u(x)|^p dx + \frac{1}{p} \int_{\mathbb{R}^n} |u(x)|^p dx : u \in W^{1,p}(\mathbb{R}^n), \|u\|_{L^s(\mathbb{R}^n)} = 1 \right\}, \quad (2.3)$$

Since u_∞ is a minimizer of (2.3), then u_∞ satisfies

$$-\Delta_p u_\infty + |u_\infty|^{p-2} u_\infty = \lambda |u_\infty|^{s-2} u_\infty,$$

where

$$\lambda = pE(u_\infty), \quad (2.4)$$

is the so called Lagrange multiplier. By the Pohozaev identity[9], we have

$$\int_{\mathbb{R}^n} |u_\infty|^p dx = \frac{p}{n} \int_{\mathbb{R}^n} |\nabla u_\infty|^p dx = \frac{\lambda p}{n+p} \int_{\mathbb{R}^n} |u_\infty|^s dx. \quad (2.5)$$

Let

$$v(x) = \left(\frac{p\lambda}{n}\right)^{\frac{n}{p^2}} u_\infty\left(\left(\frac{p}{n}\right)^{\frac{1}{p}} x\right), \quad (2.6)$$

then v satisfies (1.3). By the definition of ground state and (1.9), it follows from (2.6) that

$$\lambda^{\frac{n}{p}} \int_{\mathbb{R}^n} |u_\infty|^p dx = \int_{\mathbb{R}^n} |v|^p dx \geq a^*{}^{\frac{n}{p}}. \quad (2.7)$$

Hence, (2.5) and (2.7) together with $\int_{\mathbb{R}^n} |u_\infty|^s = 1$ imply that

$$\lambda \geq \left(\frac{n+p}{p}\right)^{\frac{p}{n+p}} a^*{}^{\frac{n}{n+p}}. \quad (2.8)$$

So, (2.1) holds by using (2.2), (2.4) and (2.8).

Next, we claim that any $Q \in \mathcal{G}$ is an extremal function of (2.1).

Indeed, if $Q \in \mathcal{G}$ then Q is a ground state of (1.3), and (1.6) and (1.9) hold, that is,

$$\frac{n}{n+p} \int_{\mathbb{R}^n} |Q|^s dx = \int_{\mathbb{R}^n} |\nabla Q|^p dx = \int_{\mathbb{R}^n} |Q|^p dx = a^*{}^{\frac{n}{p}}.$$

Therefore, Q satisfies the equality of (2.1), so does $c_1 Q(c_2 x)$ for any $c_1, c_2 \in \mathbb{R} \setminus \{0\}$.

Now, let u be an extremal of (2.1), then by a similar arguments to [24] we know that u satisfies

$$-\Delta_p u + a_1 |u|^{p-2} u - a_2 |u|^{s-2} u = 0,$$

for some $a_1, a_2 > 0$. Let $\hat{u}(x) = \left(\frac{pa_2}{na_1}\right)^{\frac{n}{p^2}} u\left(\left(\frac{p}{na_1}\right)^{\frac{1}{p}} x\right)$, then \hat{u} satisfies

$$-\Delta_p \hat{u} + \frac{p}{n} |\hat{u}|^{p-2} \hat{u} - |\hat{u}|^{s-2} \hat{u} = 0,$$

and \hat{u} is also an extremal of (2.1). Then by (1.6) we have

$$\int_{\mathbb{R}^n} |\hat{u}|^p = (a^*)^{\frac{n}{p}}.$$

Thus, $\hat{u} \in \mathcal{G}$ and $u = \left(\frac{na_1}{pa_2}\right)^{\frac{n}{p^2}} \hat{u}\left(\left(\frac{na_1}{p}\right)^{\frac{1}{p}} x\right)$.

□

Using Lemma 2.1 one can quickly get the following result.

Lemma 2.2 *Let $0 \leq w_0 \in W^{1,p}(\mathbb{R}^n)$ satisfy the equation*

$$-\Delta_p w_0 + \frac{p}{n} w_0^{p-1} = a^* w_0^{s-1}, \quad (2.9)$$

and

$$\int_{\mathbb{R}^n} |\nabla w_0|^p dx = \int_{\mathbb{R}^n} |w_0|^p dx = 1. \quad (2.10)$$

Then, w_0 satisfies the equality of (2.1) and $w_0 = a^{*- \frac{n}{p^2}} Q(x)$ for some $Q \in \mathcal{G}$.

Proof. By (2.9) and (2.10), it is easy to see that

$$\int_{\mathbb{R}^n} |w_0|^s dx = \frac{n+p}{na^*}, \quad (2.11)$$

using again (2.10) we know that w_0 satisfies the equality of (2.1). Then, Lemma 2.1 implies that there exist $c_1, c_2 > 0$ such that

$$w_0 = c_1 Q(c_2 x),$$

for some $Q \in \mathcal{G}$. Using (2.10) and (2.11) together with (1.6) and (1.9), we get $c_1 = a^{*- \frac{n}{p^2}}$ and $c_2 = 1$, thus the proof is completed. \square

Lemma 2.3 *Suppose $V \in L_{loc}^\infty(\mathbb{R}^n)$ with $\lim_{|x| \rightarrow \infty} V(x) = \infty$, then the embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^n)$ is compact, for any $p \leq q < p^* = \begin{cases} \frac{np}{n-p}, p < n, \\ +\infty, p \geq n. \end{cases}$*

Proof. This lemma can be proved by almost the same way as that of Lemma 5.1 in [26] or section 3 of [4], where only $p = 2$ is considered. \square

Lemma 2.4 *If u_a is a minimizer of problem (1.4), then u_a is a ground state of (1.1) for some $\mu = \mu_a$.*

Proof. Let u_a be a minimizer of problem (1.4), then there is a μ_a , i.e., the so called Lagrange multiplier, such that

$$-\Delta_p u_a + V(x)|u_a|^{p-2} u_a = \mu_a |u_a|^{p-2} u_a + a |u_a|^{s-2} u_a. \quad (2.12)$$

Define

$$J_a(u) = \frac{1}{p} \int_{\mathbb{R}^n} [|\nabla u|^p + (V(x) - \mu_a)|u|^p] dx - \frac{a}{s} \int_{\mathbb{R}^n} |u|^s dx. \quad (2.13)$$

Then, to prove u_a is a ground state of (1.1), we need only to show that $J_a(u_a) \leq J_a(v)$ for any nontrivial weak solution v of (2.12). For this purpose, let $v(x) \not\equiv 0$ be solution of (2.12), then we see that

$$\int_{\mathbb{R}^n} [|\nabla v|^p + (V(x) - \mu_a)|v|^p] dx = a \int_{\mathbb{R}^n} |v|^s dx. \quad (2.14)$$

It follows from (2.13) and (2.14) that

$$J_a(v) = \left(\frac{1}{p} - \frac{1}{s}\right)a \int_{\mathbb{R}^n} |v|^s dx = \frac{a}{n+p} \int_{\mathbb{R}^n} |v|^s dx.$$

Since u_a satisfies (2.12), then (2.14) holds also for u_a , this implies that

$$J_a(u_a) = \frac{a}{n+p} \int_{\mathbb{R}^n} |u_a|^s dx.$$

Now, we set $d = \|v\|_{L^p(\mathbb{R}^n)}$ and $\bar{v} = \frac{v(x)}{d}$, then $\|\bar{v}\|_{L^p(\mathbb{R}^n)} = 1$. Note that u_a is a minimizer of (1.4), hence

$$E_a(\bar{v}) \geq E_a(u_a), \|u_a\|_{L^p(\mathbb{R}^n)} = 1,$$

which means that

$$J_a(\bar{v}) = \frac{1}{p} E_a(\bar{v}) - \frac{\mu_a}{p} \int_{\mathbb{R}^n} |\bar{v}|^p dx \geq \frac{1}{p} E_a(u_a) - \frac{\mu_a}{p} \int_{\mathbb{R}^n} |u_a|^p dx = J_a(u_a). \quad (2.15)$$

On the other hand, by the definition of \bar{v} and J_a as well as (2.14), we see that

$$J_a(\bar{v}) = \left(\frac{1}{pd^p} - \frac{1}{sd^s}\right)a \int_{\mathbb{R}^n} |v|^s dx \leq \frac{a}{n+p} \int_{\mathbb{R}^n} |v|^s dx = J_a(v),$$

this and (2.15) show that $J_a(v) \geq J_a(u_a)$. We complete the proof. \square

Lemma 2.5 *Let $a > 0$, $b > 0$ and $p > 1$. then there exists $C_p = C(p) > 0$ such that*

$$(a+b)^p \leq a^p + b^p + C_p a^{p-1}b + C_p ab^{p-1}.$$

.

\square

3 Existence of ground states

The aim of this section is to prove Theorem 1.1. By Theorem 1.1 and Lemma 2.4 we then get the existence and non-existence of ground states of (1.1). For the $p = 2$ case we refer to [3, 10, 26].

Proof of Theorem 1.1. (i). For any $u \in \mathcal{H}$ with $\|u\|_{L^p} = 1$, by (2.1) and (V) we know that, if $a \in [0, a^*)$

$$E_a(u) \geq \left(1 - \frac{a}{a^*}\right) \int_{\mathbb{R}^n} |\nabla u|^p dx + \int_{\mathbb{R}^n} V(x) |u(x)|^p dx$$

$$\geq \left(1 - \frac{a}{a^*}\right) \int_{\mathbb{R}^n} |\nabla u|^p dx \geq 0. \quad (3.1)$$

So, $e(a)$ in (1.4) is well defined. Let $\{u_m\} \subset \mathcal{H}$ be a minimizing sequence, that is, $\|u_m\|_{L^p(\mathbb{R}^n)} = 1$ and $\lim_{m \rightarrow \infty} E_a(u_m) = e(a)$. Using (3.1), we see that both $\int_{\mathbb{R}^n} |\nabla u_m(x)|^p dx$ and $\int_{\mathbb{R}^n} V(x)|u_m(x)|^p dx$ are bounded, hence $\{u_m\}$ is bounded in \mathcal{H} . By Lemma 2.3, we can extract a subsequence such that

$$u_m \xrightarrow{m} u \text{ in } \mathcal{H} \text{ and } u_m \xrightarrow{m} u \text{ strongly in } L^q(\mathbb{R}^n), \text{ for any } p \leq q < p^*,$$

for some $u \in \mathcal{H}$. Then, $\int_{\mathbb{R}^n} |u(x)|^p = 1$ and $E_a(u) = e(a)$ by the weak lower semi-continuity of E_a . This implies u is a minimizer of $e(a)$.

(ii). Choose a non-negative $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = 1 \text{ if } |x| \leq 1, \text{ and } \varphi(x) = 0 \text{ if } |x| \geq 2.$$

For any $x_0 \in \mathbb{R}^n$, $\tau > 0$ and $R > 0$, motivated by [10] we let

$$u_\tau(x) = A_{R,\tau} \frac{\tau^{\frac{n}{p}}}{\|Q\|_{L^p}} \varphi\left(\frac{x - x_0}{R}\right) Q(\tau(x - x_0)), \quad (3.2)$$

where Q is a ground state of (1.3) and $A_{R,\tau}$ is chosen such that $\int_{\mathbb{R}^n} |u_\tau(x)|^p dx = 1$ and then $\lim_{R\tau \rightarrow \infty} A_{R,\tau} = 1$. In fact, it follows from (1.8) that

$$\begin{aligned} \frac{1}{A_{R,\tau}^p} &= \frac{\int_{x \leq \tau R} Q^p(x) dx + \int_{\tau R < |x| \leq 2\tau R} \varphi^p\left(\frac{x}{\tau R}\right) Q^p(x) dx}{\|Q\|_{L^p}^p} \\ &= 1 + O(e^{-\delta\tau R}), \quad \text{as } \tau R \rightarrow \infty. \end{aligned} \quad (3.3)$$

By Lemma 2.5 and (1.8) we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| \frac{1}{R} \nabla \varphi\left(\frac{x}{\tau R}\right) Q(x) + \tau \varphi\left(\frac{x}{\tau R}\right) \nabla Q(x) \right|^p dx - \int_{\mathbb{R}^n} \left| \tau \varphi\left(\frac{x}{\tau R}\right) \nabla Q(x) \right|^p dx \\ & \leq \int_{\mathbb{R}^n} \left| \frac{1}{R} \nabla \varphi\left(\frac{x}{\tau R}\right) Q(x) \right|^p dx + C_p \int_{\mathbb{R}^n} \left| \frac{1}{R} \nabla \varphi\left(\frac{x}{\tau R}\right) Q(x) \right|^{p-1} \left| \tau \varphi\left(\frac{x}{\tau R}\right) \nabla Q(x) \right| dx \\ & \quad + C_p \int_{\mathbb{R}^n} \left| \frac{1}{R} \nabla \varphi\left(\frac{x}{\tau R}\right) Q(x) \right| \left| \tau \varphi\left(\frac{x}{\tau R}\right) \nabla Q(x) \right|^{p-1} dx \\ & = O(e^{-\delta\tau R}), \quad \text{as } \tau R \rightarrow \infty. \end{aligned}$$

Then, by (3.3) and the exponential decay of Q (1.8), we see that

$$\int_{\mathbb{R}^n} |\nabla u_\tau(x)|^p dx - \frac{na}{n+p} \int_{\mathbb{R}^n} |u_\tau(x)|^s dx$$

$$\begin{aligned}
&= \frac{A_{R,\tau}^p \tau^n}{\|Q\|_{L^p}^p} \int_{\mathbb{R}^n} \left| \frac{1}{R} \nabla \varphi\left(\frac{x-x_0}{R}\right) Q(\tau(x-x_0)) + \varphi\left(\frac{x-x_0}{R}\right) \nabla Q(\tau(x-x_0)) \tau \right|^p dx \\
&\quad - \frac{na}{n+p} \frac{A_{R,\tau}^s \tau^{\frac{sn}{p}}}{\|Q\|_{L^p}^s} \int_{\mathbb{R}^n} \left| \varphi\left(\frac{x-x_0}{R}\right) Q(\tau(x-x_0)) \right|^s dx \\
&= \frac{A_{R,\tau}^p}{\|Q\|_{L^p}^p} \int_{\mathbb{R}^n} \left| \frac{1}{R} \nabla \varphi\left(\frac{x}{\tau R}\right) Q(x) + \tau \varphi\left(\frac{x}{\tau R}\right) \nabla Q(x) \right|^p dx \\
&\quad - \frac{na}{n+p} \frac{A_{R,\tau}^s \tau^p}{\|Q\|_{L^p}^s} \int_{\mathbb{R}^n} \left| \varphi\left(\frac{x}{\tau R}\right) Q(x) \right|^s dx \\
&\leq \frac{A_{R,\tau}^p}{\|Q\|_{L^p}^p} \int_{\mathbb{R}^n} \left| \tau \varphi\left(\frac{x}{\tau R}\right) \nabla Q(x) \right|^p dx - \frac{na}{n+p} \frac{A_{R,\tau}^s \tau^p}{\|Q\|_{L^p}^s} \int_{\mathbb{R}^n} \left| \varphi\left(\frac{x}{\tau R}\right) Q(x) \right|^s dx \\
&\quad + O(e^{-\delta\tau R}) \text{ as } \tau R \rightarrow \infty \\
&\leq \frac{\tau^p}{\|Q\|_{L^p}^p} \left(\int_{\mathbb{R}^n} |\nabla Q|^p dx - \frac{na}{(n+p)a^*} \int_{\mathbb{R}^n} |Q|^s dx \right) + O(e^{-\delta\tau R}), \text{ as } R\tau \rightarrow \infty \quad (3.4)
\end{aligned}$$

It then follows from (3.4) and (1.6) that

$$\int_{\mathbb{R}^n} |\nabla u_\tau|^p dx - \frac{na}{n+p} \int_{\mathbb{R}^n} |u_\tau|^s dx \leq \tau^p \left(1 - \frac{a}{a^*}\right) + O(e^{-\delta\tau R}). \quad (3.5)$$

On the other hand, since $u_\tau(x)$ is bounded and has compact support, the convergence

$$\lim_{\tau \rightarrow \infty} \int_{\mathbb{R}^n} V(x) |u_\tau(x)|^p dx = \lim_{\tau \rightarrow \infty} \int_{\mathbb{R}^n} \frac{V\left(\frac{x}{\tau} + x_0\right)}{\|Q\|_{L^p}^p} \varphi^p\left(\frac{x}{\tau R}\right) Q^p(x) dx = V(x_0) \quad (3.6)$$

holds for all $x_0 \in \mathbb{R}^n$.

When $a > a^*$, it follows from (3.5) and (3.6) that

$$e(a) \leq \lim_{\tau \rightarrow \infty} E_a(u) = -\infty.$$

This implies that for any $a > a^*$, $e(a)$ is unbounded from below, and the nonexistence of minimizers is therefore proved.

When $a = a^*$, taking $x_0 \in \mathbb{R}^n$ such that $V(x_0) = 0$, then (3.5) and (3.6) imply that $e(a^*) \leq 0$, but we know that $e(a^*) \geq 0$ by (3.1), so $e(a^*) = 0$. If there exists a minimizer $u_0 \in \mathcal{H}$ for $e(a^*) = 0$ with $\|u_0\|_{L^p} = 1$, then

$$\int_{\mathbb{R}^n} V(x) |u_0(x)|^p dx = \inf_{x \in \mathbb{R}^n} V(x) = 0,$$

and

$$\int_{\mathbb{R}^n} |\nabla u_0(x)|^p dx = \frac{na^*}{n+p} \int_{\mathbb{R}^n} |u_0(x)|^s dx.$$

These lead to a contradiction, since the first equality implies that u_0 must have compact support, while the second equality means that u_0 has to be a nonnegative ground state of (1.3) by Lemma 2.1, thus $u_0 > 0$ by the strong maximum principle [22]. So problem (1.4) has no minimizer for $a = a^*$.

Note that (3.1) implies that $e(a) > 0$ for $a < a^*$. We have already shown that $e(a^*) = 0$ and $e(a) = -\infty$ if $a > a^*$, hence it remains to prove that $\lim_{a \nearrow a^*} e(a) = 0$.

Indeed, let $x_0 \in \mathbb{R}^n$ be such that $V(x_0) = 0$, set $\tau = (a^* - a)^{-\frac{1}{p+1}}$. Then if $a \nearrow a^*$, it follows easily from (3.5) and (3.6) that $\limsup_{a \nearrow a^*} e(a) \leq 0$, hence $\lim_{a \nearrow a^*} e(a) = 0$. \square

4 Blowup behavior for general trapping potential

In this section, we come to analyze the concentration (blowup) behavior of the ground states of (1.1) as $a \nearrow a^*$ under the general assumption (V), that is, to give a proof of Theorem 1.2.

Let u_a be a nonnegative minimizer of (1.4), thus u_a satisfies the following equation

$$-\Delta_p u_a + V(x)u_a^{p-1} = \mu_a u_a^{p-1} + a u_a^{s-1}, \quad (4.1)$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier.

Proof of Theorem 1.2.

(i). By contradiction, if (1.10) is false, then there exists a sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \rightarrow \infty$ such that $\{u_{a_k}(x)\}$ is bounded in \mathcal{H} . By applying Lemma 2.3, there exist a subsequence of $\{a_k\}$ (still denoted by $\{a_k\}$) and $u_0 \in \mathcal{H}$ such that

$$u_{a_k} \xrightarrow{k} u_0 \text{ weakly in } \mathcal{H} \text{ and } u_{a_k} \xrightarrow{k} u_0 \text{ in } L^p(\mathbb{R}^n).$$

Thus,

$$0 = e(a^*) \leq E_{a^*}(u_0) \leq \lim_{k \rightarrow \infty} E_{a_k}(u_{a_k}) = \lim_{k \rightarrow \infty} e(a_k) = 0,$$

since $e(a) \rightarrow 0$ as $a \nearrow a^*$, by Theorem 1.1. This shows that u_0 is a minimizer of $e(a^*)$, which is impossible by Theorem 1.1(ii). So, part (i) is proved.

(ii). For any solution u_a of (4.1), by the result of [15] we know that $u_a \in C_{loc}^{1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$ and

$$u_a(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

this implies that each u_a has at least one maximum point. Let \bar{z}_a be a global maximum point and define

$$\bar{w}_a(x) = \varepsilon_a^{\frac{n}{p}} u_a(\varepsilon_a x + \bar{z}_a), \quad (4.2)$$

then

$$\int_{\mathbb{R}^n} |\nabla \bar{w}_a|^p = \int_{\mathbb{R}^n} |\bar{w}_a|^p = 1. \quad (4.3)$$

By (2.1), we know that

$$0 \leq \int_{\mathbb{R}^n} |\nabla u_a|^p - \frac{n}{n+p} a \int_{\mathbb{R}^n} |u_a|^s = \varepsilon_a^{-p} - \frac{n}{n+p} a \int_{\mathbb{R}^n} |u_a|^s \leq e(a).$$

By part (i) and Theorem 1.1 (ii), we have

$$\varepsilon_a \rightarrow 0 \quad \text{and} \quad e(a) \rightarrow 0 \quad \text{as} \quad a \nearrow a^*,$$

then

$$\int_{\mathbb{R}^n} |\bar{w}_a|^s = \varepsilon_a^p \int_{\mathbb{R}^n} |u_a|^s \rightarrow \frac{n+p}{na^*} \quad \text{as} \quad a \nearrow a^*. \quad (4.4)$$

Now, we claim that

$$\liminf_{a \nearrow a^*} \int_{B_2(0)} |\bar{w}_a|^p \geq \eta > 0. \quad (4.5)$$

Indeed, it follows from (4.1) that

$$\mu_a = e(a) - \frac{pa}{n+p} \int_{\mathbb{R}^n} |u_a|^s,$$

this together with (4.4) indicates that

$$\varepsilon_a^p \mu_a \rightarrow -\frac{p}{n} \quad \text{as} \quad a \nearrow a^*. \quad (4.6)$$

Moreover, by (4.1) and (4.2), \bar{w}_a satisfies that

$$-\Delta_p \bar{w}_a(x) + \varepsilon_a^p V(\varepsilon x + \bar{z}_a) \bar{w}_a^{p-1}(x) = \varepsilon_a^p \mu_a \bar{w}_a^{p-1}(x) + a \bar{w}_a^{s-1}(x), \quad (4.7)$$

and since $\bar{w}_a \geq 0$ and $\varepsilon_a^p \mu_a \leq 0$ for a close to a^* , it follows from (4.7) that

$$-\Delta_p \bar{w}_a - c(x) \bar{w}_a^{p-1} \leq 0, \quad \text{where} \quad c(x) = a \bar{w}_a^{s-p}.$$

Thus by Theorem 7.1.1 of [21] we have

$$\sup_{B_1(\xi)} \bar{w}_a \leq C \left(\int_{B_2(\xi)} |\bar{w}_a|^p \right)^{\frac{1}{p}}, \quad (4.8)$$

where ξ is an arbitrary point in \mathbb{R}^n and $C > 0$ depends only on the upper bound of $\|c(x)\|_{L^{\frac{p}{p(1-\epsilon)}}(B_2(\xi))}$, i.e., $\|\bar{w}_a\|_{L^{\frac{p}{1-\epsilon}}(B_2(\xi))}$, for some $0 < \epsilon \leq 1$. On the other hand,

$$\bar{w}_a(0) \geq \zeta \quad \text{uniformly as} \quad a \nearrow a^* \quad \text{for some} \quad \zeta > 0, \quad (4.9)$$

since 0 is a global maximum point of \bar{w}_a . Otherwise, there exists a sequence $a_k \nearrow a^*$ such that $\bar{w}_{a_k}(0) = \|\bar{w}_{a_k}\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{k} 0$, then by concentration-compactness lemma [18] we have $\int_{\mathbb{R}^n} |\bar{w}_{a_k}|^s \rightarrow 0$ as $k \rightarrow \infty$, which contradicts (4.4). So, (4.8) and (4.9) imply (4.5).

In what follows, we come to prove (1.11) by using (4.5). Since (1.5) and (2.1), we have

$$\int_{\mathbb{R}^n} V(x) |u_a|^p dx \leq e(a) \rightarrow 0 \quad \text{as } a \nearrow a^*,$$

that is

$$\int_{\mathbb{R}^n} V(x) |u_a|^p dx = \int_{\mathbb{R}^n} V(\varepsilon_a x + \bar{z}_a) |\bar{w}_a|^p dx \rightarrow 0 \quad \text{as } a \nearrow a^*. \quad (4.10)$$

By contradiction, if (1.11) is false, then there is a constant $\delta > 0$ and a sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \rightarrow \infty$ such that

$$\varepsilon_{a_k} \xrightarrow{k} 0 \quad \text{and} \quad \liminf_{k \rightarrow \infty} \text{dist}(\bar{z}_{a_k}, \mathcal{A}) \geq \delta > 0, \quad (4.11)$$

then, there exists $C_\delta > 0$ such that

$$\liminf_{k \rightarrow \infty} V(\bar{z}_{a_k}) \geq 2C_\delta. \quad (4.12)$$

Indeed, suppose such C_δ does not exist, then, up to a subsequence, there exist $\{\bar{z}_{a_k}\} \subset \mathbb{R}^n$ such that $V(\bar{z}_{a_k}) \xrightarrow{k} 0$. Since $V(x)$ satisfies (V), we have $\{\bar{z}_{a_k}\}$ is bounded and thus $\bar{z}_{a_k} \xrightarrow{k} z_0$, for some $z_0 \in \mathbb{R}^n$. By the continuity of $V(x)$ we have $z_0 \in \mathcal{A}$, but this contradicts (4.11), so (4.12) is proved. By Fatou's Lemma and (4.5), we see that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} V(\varepsilon_{a_k} x + \bar{z}_{a_k}) |\bar{w}_{a_k}|^p dx \geq \int_{\mathbb{R}^n} \lim_{k \rightarrow \infty} V(\varepsilon_{a_k} x + \bar{z}_{a_k}) |\bar{w}_{a_k}|^p dx \geq C_\delta \eta,$$

which contradicts (4.10). Therefore, (1.11) holds.

(iii). Let $\{a_k\}$ be a sequence such that $a_k \nearrow a^*$ as $k \rightarrow \infty$. For simplicity, we set

$$u_k(x) := u_{a_k}(x), \quad \bar{w}_k := \bar{w}_{a_k} \geq 0, \quad \bar{z}_k := \bar{z}_{a_k} \quad \text{and} \quad \varepsilon_k := \varepsilon_{a_k} > 0.$$

By (1.11), (4.3) and (4.4), there exists a subsequence of $\{a_k\}$, still denoted by $\{a_k\}$, such that

$$\lim_{k \rightarrow \infty} \bar{z}_k = x_0 \quad \text{with} \quad V(x_0) = 0,$$

and

$$\bar{w}_k \xrightarrow{k} w_0 \geq 0 \quad \text{weakly in } W^{1,p}(\mathbb{R}^n)$$

for some $w_0 \in W^{1,p}(\mathbb{R}^n)$. Moreover, \bar{w}_k satisfies

$$-\Delta_p \bar{w}_k(x) + \varepsilon_k^p V(\varepsilon_k x + \bar{z}_k) \bar{w}_k^{p-1}(x) = \varepsilon_k^p \mu_k \bar{w}_k^{p-1}(x) + a_k \bar{w}_k^{s-1}(x).$$

Motivated by the idea of [25, 14], we claim that

$$\bar{w}_k \xrightarrow{k} w_0 \quad \text{strongly in } W^{1,p}(\mathbb{R}^n). \quad (4.13)$$

Indeed, by Theorem 1.1 and the definition of \bar{w}_k we have

$$0 \leq \lim_{k \rightarrow \infty} \left(\varepsilon_k^{-p} \int_{\mathbb{R}^n} |\nabla \bar{w}_k|^p - \frac{na_k}{n+p} \varepsilon_k^{-p} \int_{\mathbb{R}^n} |\bar{w}_k|^s \right) \leq \lim_{k \rightarrow \infty} e(a_k) = 0, \quad (4.14)$$

and thus by (4.3) and (4.14) that

$$\int_{\mathbb{R}^n} |\bar{w}_k|^p \equiv 1,$$

and

$$\lim_{k \rightarrow \infty} \left(\int_{\mathbb{R}^n} |\nabla \bar{w}_k|^p - \frac{na^*}{n+p} \int_{\mathbb{R}^n} |\bar{w}_k|^s \right) = 0. \quad (4.15)$$

To prove (4.13), we show first that

$$\bar{w}_k \xrightarrow{k} w_0 \quad \text{strongly in } L^p(\mathbb{R}^n). \quad (4.16)$$

By Lemma III.1 of [18], to show (4.16) we only need to exclude the “vanishing case” and “dichotomy case” of the function sequence $\{|\bar{w}_k|^p\}$. In fact, we can easily rule out the “vanishing case” by (4.5). Then, arguing indirectly, suppose the “dichotomy case” occurs and taking (4.5) into consideration, there exists $0 < \rho < 1$ such that for any $\varepsilon > 0$ there are $R_k > 0$, $\{y_k\} \subset \mathbb{R}^n$ and two function sequences $\{w_k^1\}, \{w_k^2\} \subset W^{1,p}(\mathbb{R}^n)$ with $\text{Supp } w_k^1 \subset B_{R_k}(y_k)$ and $\text{Supp } w_k^2 \subset \mathbb{R}^n \setminus B_{2R_k}(y_k)$ satisfy

$$\left| \int_{B_{R_k}(y_k)} |w_k^1|^p - \rho \right| \leq \varepsilon, \quad \left| \int_{\mathbb{R}^n \setminus B_{2R_k}(y_k)} |w_k^2|^p - (1 - \rho) \right| \leq \varepsilon; \quad (4.17)$$

and, up to a subsequence,

$$\int_{\mathbb{R}^n} |\bar{w}_k - (w_k^1 + w_k^2)|^s \leq C_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4.18)$$

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \left(|\nabla \bar{w}_k|^p - |\nabla w_k^1|^p - |\nabla w_k^2|^p \right) \geq 0. \quad (4.19)$$

Then, by (4.17)-(4.19), (2.1) and (4.4) we have

$$\left(\int_{\mathbb{R}^n} |\nabla \bar{w}_k|^p - \frac{na^*}{n+p} \int_{\mathbb{R}^n} |\bar{w}_k|^s \right)$$

$$\begin{aligned}
&\geq \int_{B_{R_k}(y_k)} \left(|\nabla w_k^1|^p - \frac{na^*}{n+p} |w_k^1|^s \right) + \int_{\mathbb{R}^n \setminus B_{2R_k}(y_k)} \left(|\nabla w_k^2|^p - \frac{na^*}{n+p} |w_k^2|^s \right) - C_\varepsilon \\
&\geq \frac{na^*}{n+p} \left(((\rho + \varepsilon)^{-\frac{p}{n}} - 1) \int_{\mathbb{R}^n} |w_k^1|^s + ((1 - \rho + \varepsilon)^{-\frac{p}{n}} - 1) \int_{\mathbb{R}^n} |w_k^2|^s \right) - C_\varepsilon \\
&> 0,
\end{aligned} \tag{4.20}$$

since ε can be arbitrarily small. Then (4.20) leads to a contradiction with (4.15). So, (4.16) holds. (4.16) and the boundedness of $\{\bar{w}_k\}$ in $W^{1,p}(\mathbb{R}^n)$ imply that

$$\bar{w}_k \xrightarrow{k} w_0 \quad \text{strongly in } L^s(\mathbb{R}^n). \tag{4.21}$$

Combine (4.16)(4.21) with (2.1) and (4.4), we have $\int_{\mathbb{R}^n} |\nabla w_0|^p \geq 1$ and thus

$$\int_{\mathbb{R}^n} |\nabla w_0|^p = 1,$$

by the weak lower semi-continuity of the norm. Since $W^{1,p}(\mathbb{R}^n)$ is a uniformly convex Banach space [1], (4.13) holds by [5]. Moreover, w_0 satisfies

$$-\Delta_p w_0 + \frac{p}{n} w_0^{p-1} = a^* w_0^{s-1},$$

and

$$\int_{\mathbb{R}^n} |\nabla w_0|^p = \int_{\mathbb{R}^n} |w_0|^p = 1.$$

So, Lemma 2.2 implies that

$$w_0 = \frac{Q(x)}{a^* \frac{n}{p^2}}, \tag{4.22}$$

for some $Q \in \mathcal{G}$. Then (4.13) and (4.22) gives the proof of Theorem 1.2. \square

5 Refined blowup behavior for polynomial type potential

This section is devoted to prove Theorem 1.3. In what follows, we always denote $\{a_k\}$ to be the convergent subsequence in Theorem 1.2. Our first lemma is to address the upper bound of $e(a_k)$.

Lemma 5.1 *If $V(x)$ satisfies (1.2), then*

$$0 \leq e(a_k) \leq \frac{(a^* - a_k)^{\frac{q}{p+q}}}{a^* \frac{n+q}{p+q}} \lambda^{\frac{p}{p+q}} \left(\left(\frac{q}{p} \right)^{\frac{p}{p+q}} + \left(\frac{p}{q} \right)^{\frac{q}{p+q}} + o(1) \right) \quad \text{as } k \rightarrow \infty, \tag{5.1}$$

where λ is given by (1.14) and $o(1)$ is a quantity depends only on k .

Proof. Let $Q \in \mathcal{G}$ be given in Theorem 1.2 and y_0 as in (1.13). Take

$$u(x) = A_{R,\tau} \frac{\tau^{\frac{n}{p}}}{\|Q\|_{L^p}} \varphi\left(\frac{x-x_0}{R}\right) Q(\tau(x-x_0)-y_0)$$

with $x_0 \in \mathcal{Z}$ as a trial function. Let $R = \frac{1}{\sqrt{\tau}}$ and it is easy to see $R \rightarrow 0$ and $\tau R \rightarrow \infty$ as $\tau \rightarrow \infty$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} V(x)|u|^p &= A_{R,\tau}^p \int_{\mathbb{R}^n} V(x) \frac{\tau^n}{\|Q\|_{L^p}^p} \varphi^p\left(\frac{x-x_0}{R}\right) Q^p(\tau(x-x_0)-y_0) dx \\ &= \frac{A_{R,\tau}^p}{\|Q\|_{L^p}^p} \int_{\mathbb{R}^n} \tau^n |x-x_0|^q \varphi^p\left(\frac{x-x_0}{R}\right) Q^p(\tau(x-x_0)-y_0) \frac{V(x)}{|x-x_0|^q} dx \\ &\leq \frac{A_{R,\tau}^p}{\|Q\|_{L^p}^p} \tau^{-q} \left(\int_{\mathbb{R}^n} |x|^q Q^p(x-y_0) dx \lim_{x \rightarrow x_0} \frac{V(x)}{|x-x_0|^q} + o(1) \right) \\ &\leq A_{R,\tau}^p a^{*- \frac{n}{p}} \tau^{-q} (\lambda + o(1)), \end{aligned} \quad (5.2)$$

with $o(1) \rightarrow 0$ as $\tau \rightarrow \infty$. By (5.2), (3.3) and (3.5), we see that, for large τ ,

$$\begin{aligned} e(a_k) &\leq E_{a_k}(u) = \int_{\mathbb{R}^n} |\nabla u|^p - \frac{n}{n+p} a_k \int_{\mathbb{R}^n} |u|^s + \int_{\mathbb{R}^n} V(x)|u|^p \\ &\leq \frac{a^* - a_k}{a^*} \tau^p + a^{*- \frac{n}{p}} \tau^{-q} (\lambda + o(1)) + O(e^{-\delta R \tau}). \end{aligned}$$

Choose $\tau = (\frac{a^{*1-\frac{n}{p}} \lambda q}{(a^* - a_k)p})^{\frac{1}{p+q}}$ and thus $\tau \xrightarrow{k} \infty$, then

$$0 \leq e(a_k) \leq \frac{(a^* - a_k)^{\frac{q}{p+q}}}{a^{* \frac{n+q}{p+q}}} \lambda^{\frac{p}{p+q}} \left(\left(\frac{q}{p}\right)^{\frac{p}{p+q}} + \left(\frac{p}{q}\right)^{\frac{q}{p+q}} + o(1) \right) \quad \text{as } k \rightarrow \infty.$$

□

Based on Lemma 5.1 and Theorem 1.2, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3.

(i). From (2.1) we have

$$\begin{aligned} e(a_k) = E_{a_k}(u_{a_k}) &= \int_{\mathbb{R}^n} |\nabla u_{a_k}|^p - \frac{n}{n+p} a \int_{\mathbb{R}^n} |u_{a_k}|^s + \int_{\mathbb{R}^n} V(x)|u_{a_k}|^p \\ &\geq \frac{a^* - a_k}{a^*} \int_{\mathbb{R}^n} |\nabla u_{a_k}|^p + \int_{\mathbb{R}^n} V(x)|u_{a_k}|^p \\ &= \frac{a^* - a_k}{a^*} \varepsilon_{a_k}^{-p} + \int_{\mathbb{R}^n} V(\varepsilon_{a_k} x + \bar{z}_{a_k}) |\bar{w}_{a_k}|^p dx, \end{aligned}$$

where \bar{w}_{a_k} is given by (4.2). By Theorem 1.2, we may assume that $\bar{z}_{a_k} \rightarrow x_i$ with $x_i \in \mathcal{A}$ as $k \rightarrow \infty$. Define

$$\bar{V}_{a_k}(x) = \frac{V(\varepsilon_{a_k} x + \bar{z}_{a_k})}{c_i |\varepsilon_{a_k} x + \bar{z}_{a_k} - x_i|^{q_i}} \quad \text{with } c_i = \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^{q_i}},$$

then $\bar{V}_{a_k}(x) \rightarrow 1$ a.e. in $x \in \mathbb{R}^n$ as $k \rightarrow \infty$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} V(\varepsilon_{a_k}x + \bar{z}_{a_k})|\bar{w}_{a_k}|^p dx &= \int_{\mathbb{R}^n} \bar{V}_{a_k} c_i |\varepsilon_{a_k}x + \bar{z}_{a_k} - x_i|^{q_i} \bar{w}_{a_k}^p dx \\ &= \varepsilon_{a_k}^{q_i} \int_{\mathbb{R}^n} \bar{V}_{a_k} c_i |x + \frac{\bar{z}_{a_k} - x_i}{\varepsilon_{a_k}}|^{q_i} \bar{w}_{a_k}^p dx. \end{aligned}$$

We claim that $\limsup_{k \rightarrow \infty} \frac{|\bar{z}_{a_k} - x_i|}{\varepsilon_{a_k}} < \infty$ and $q_i = q$. Indeed, let

$$\rho_{a_k} := \int_{\mathbb{R}^n} \bar{V}_{a_k} c_i |x + \frac{\bar{z}_{a_k} - x_i}{\varepsilon_{a_k}}|^{q_i} \bar{w}_{a_k}^p dx,$$

then $\rho_{a_k} \rightarrow \infty$ if $\frac{|\bar{z}_{a_k} - x_i|}{\varepsilon_{a_k}} \rightarrow \infty$. Moreover we have

$$\begin{aligned} e(a_k) &\geq \frac{a^* - a_k}{a^*} \varepsilon_{a_k}^{-p} + \rho_{a_k} \varepsilon_{a_k}^{q_i} \\ &\geq \left(\frac{a^* - a_k}{a^*} \right)^{\frac{q_i}{p+q_i}} \rho_{a_k}^{\frac{q_i}{p+q_i}} \left(\left(\frac{q}{p} \right)^{\frac{p}{p+q}} + \left(\frac{p}{q} \right)^{\frac{q}{p+q}} \right), \end{aligned} \quad (5.3)$$

which contradicts Lemma 5.1 if $\frac{|\bar{z}_{a_k} - x_i|}{\varepsilon_{a_k}} \xrightarrow{k} \infty$. Thus

$$\limsup_{k \rightarrow \infty} \frac{|\bar{z}_{a_k} - x_i|}{\varepsilon_{a_k}} < \infty.$$

By Fatou's Lemma and the above fact, we see that

$$\liminf_{k \rightarrow \infty} \rho_{a_k} > 0,$$

the above inequality together with (5.3) and Lemma 5.1 indicates that $q_i = q$. Then using Fatou's Lemma again,

$$\begin{aligned} &\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \bar{V}_{a_k} |x + \frac{\bar{z}_{a_k} - x_i}{\varepsilon_{a_k}}|^q w_{a_k}^p dx \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \\ &\geq \frac{1}{a^{*\frac{n}{p}}} \int_{\mathbb{R}^n} |x + y_0|^q Q^p dx \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \geq \frac{1}{a^{*\frac{n}{p}}} \lambda, \end{aligned} \quad (5.4)$$

where (1.13) and (1.14) were used in the last two inequalities. So,

$$\begin{aligned} e(a_k) &\geq \frac{a^* - a_k}{a^*} \varepsilon_{a_k}^{-p} + \varepsilon_{a_k}^q \frac{1}{a^{*\frac{n}{p}}} \lambda \\ &\geq \frac{(a^* - a_k)^{\frac{q}{p+q}}}{a^{*\frac{n+q}{p+q}}} \lambda^{\frac{p}{p+q}} \left(\left(\frac{q}{p} \right)^{\frac{p}{p+q}} + \left(\frac{p}{q} \right)^{\frac{q}{p+q}} \right), \end{aligned} \quad (5.5)$$

where the equality in the last inequality holds if and only if

$$\varepsilon_{a_k} = a^{*\frac{n-p}{p(p+q)}} (a^* - a_k)^{\frac{1}{p+q}} \lambda^{-\frac{1}{p+q}} \left(\frac{p}{q}\right)^{\frac{1}{p+q}}.$$

This together with Lemma 5.1 shows Theorem 1.3(i).

(ii). By Lemma 5.1 we know that the inequality (5.4) is in fact an equality which implies $x_0 \in \mathcal{Z}$. Then we need only to show that

$$\lim_{k \rightarrow \infty} \frac{\varepsilon_k}{\sigma_k} = 1. \quad (5.6)$$

If (5.6) is false, then there exists a subsequence of $\{k\}$, still denoted by $\{k\}$, such that

$$\lim_{k \rightarrow \infty} \frac{\varepsilon_k}{\sigma_k} = \theta \neq 1, \quad \text{with } 0 \leq \theta \leq \infty.$$

From (5.5) we have

$$\begin{aligned} e(a_k) &\geq \frac{a^* - a_k}{a^*} \varepsilon_k^{-p} + \varepsilon_k^q \frac{1}{a^{*\frac{n}{p}}} \lambda \\ &\geq \frac{(a^* - a_k)^{\frac{q}{p+q}}}{a^{*\frac{n+q}{p+q}}} \lambda^{\frac{p}{p+q}} \left(\left(\frac{q}{p}\right)^{\frac{p}{p+q}} \theta^{-p} + \left(\frac{p}{q}\right)^{\frac{q}{p+q}} \theta^q \right) \\ &> \frac{(a^* - a_k)^{\frac{q}{p+q}}}{a^{*\frac{n+q}{p+q}}} \lambda^{\frac{p}{p+q}} \left(\left(\frac{q}{p}\right)^{\frac{p}{p+q}} + \left(\frac{p}{q}\right)^{\frac{q}{p+q}} \right), \end{aligned}$$

when $\lim_{k \rightarrow \infty} \frac{\varepsilon_k}{\sigma_k} = \theta \neq 1$, and this contradicts Lemma 5.1. (5.6) together with (1.12) gives (1.17) and (1.16). So the proof is complete. \square

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